Stochastic Inventory Models

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Newsboy Problem

h: inventory holding cost

b: lost sales (backlog) cost

D: demand (random variable with non-negative and continuous distribution) whose distribution function is

$$F(x) = \Pr\{D \leq x\}$$

and its density function is

$$f(x) = \frac{\partial F(x)}{\partial x}.$$

Expected cost $C(s)$

Expected cost $C(s)$ when the initial inventory is s.

$$C(s) = \mathsf{E}\left[h[s-D]^+ + b[s-D]^-\right]$$

where

$$(\cdot)^+ = \max\{\cdot, 0\} \quad (\cdot)^- = \max\{-\cdot, 0\}.$$

$$\begin{aligned} C(s) &= h\int_0^\infty \max\{s-x, 0\} f(x)dx + \\ & \quad b\int_0^\infty \max\{x-s, 0\} f(x)dx \\ &= h\int_0^s (s-x)f(x)dx + b\int_s^\infty (x-s)f(x)dx \end{aligned}$$

Optimal soluition

$$\frac{\partial C(s)}{\partial s} = h\int_0^s 1f(x)dx + b\int_s^\infty (-1)f(x)dx = hF(s) - b(1 - F(s))$$

$$\frac{\partial^2 C(s)}{\partial s^2} = (h+b)f(s)\ (>0)$$

$C(s)$ is a convex function;
$\partial C(s)/\partial s = hF(s) - b(1 - F(s)) = 0$

$$F(s^*) = \frac{b}{b+h}$$

Series local inventory model

Series inventory system with n inventory points.
Numbered $1, 2, \cdots, n$ from the demand point to the supply point.
Assume that the $n+1$-th point is an outlier supplier with unlimited inventories.

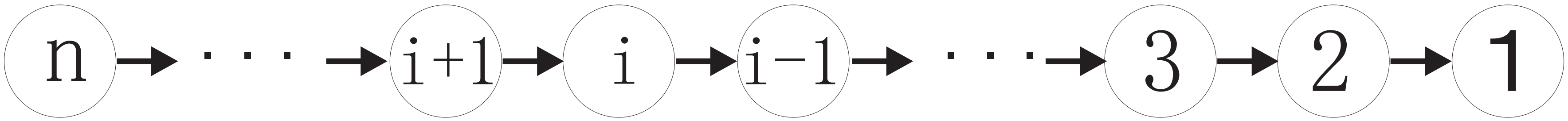

Figure 1: Series model.

Notation (1)

t: time

$I'_i(t)$: local inventory at the i-th inventory point

$B'_i(t)$: local backorders at the i-th inventory point

$IN'_i(t)$: local net inventory at the i-th point defined by

$$IN'_i(t) = I'_i(t) - B'_i(t).$$

We get:

$$I'_i(t) = [IN'_i(t)]^+$$

$$B'_i(t) = [IN'_i(t)]^-$$

Notation (2)

$IO_i(t)$: inventory on-order at the i-th point

$IT_i(t)$: inventory in transit at the i-th point

$$IO_i(t) - IT_i(t) = B'_{i+1}(t)$$

$IOP'_i(t)$: local inventory-order position at the i-th point

$$IOP'_i(t) = IN'_i(t) + IO_i(t)$$

$ITP'_i(t)$: local inventory-in-transit position at the i-th point

$$ITP'_i(t) = IN'_i(t) + IT_i(t)$$

$$IOP'_i(t) - ITP'_i(t) = B'_{i+1}(t)$$

Notation (3)

L_i'**:** lead time at the i-th inventory point

$D(s,t]$**:** demand in interval $(s,t]$

s_i'**:** local base-stock level for the i-th inventory point

b**:** backorder cost rate at the 1-th inventory point

h_i'**:** inventory holding cost rate at the i-th inventory point

Flow conservation equation

Flow conservation equation:

$$IN_i'(t + L_i') = ITP_i'(t) - D(t, t + L_i']$$

Since

$$IOP_i'(t) - ITP_i'(t) = B_{i+1}'(t),$$

we get

$$IN_i'(t + L_i') = s_i' - B_{i+1}'(t) - D(t, t + L_i'].$$

Stationary recursive equation

Let D_i be the expected value of the (stationary) lead time demand $D(t, t + L'_i]$ for the i-th point.
Since $B'_i(t) = [IN'_i(t)]^-$,

$$B'_{n+1} = 0$$

$$B'_i = \left[s'_i - B'_{i+1} - D_i\right]^-$$

Echelon-based model

$I_i(t)$: echelon inventory at the i-th inventory point, i.e.,

$$I_i(t) = I'_i(t) + \sum_{j<i} \{IT_j(t) + I'_j(t)\}$$

$B(t)$: system backorders, i.e.,

$$B(t) = B'_1(t)$$

$IN_i(t)$: echelon net inventory at the i-th point defined by

$$IN_i(t) = I_i(t) - B(t)$$

Notation (Continued)

$IOP_i(t)$: echelon inventory-order position at the i-th point

$$IOP_i(t) = IN_i(t) + IO_i(t)$$

$ITP_i(t)$: echelon inventory-in-transit position at the i-th point

$$ITP_i(t) = IN_i(t) + IT_i(t)$$

s_i: echelon base-stock level for the i-th point

Flow conservation equation

h_i: echelon inventory cost rate at the i-th point, i.e.,

$$h_i = h'_i - h'_{i+1}$$

Flow conservation equation using echelon stocks:

$$IN_i(t + L'_i) = ITP_i(t) - D(t, t + L'_i]$$

$$ITP_i(t) = \min\{s_i, IN_{i+1}(t)\}$$

Stationary solution

D_i : stationary value of $D(t, t + L'_i]$

$$ITP_n = s_n$$

$$IN_i = ITP_i - D_i$$

$$ITP_i = \min\{s_i, IN_{i+1}\}$$

Objective function

Local inventory model

$$\mathsf{E}\left[\sum_{i=1}^{n} h'_i I'_i + \sum_{i=2}^{n} h'_i IT_{i-1} + bB'_1\right]$$

Echelon inventory model

$$\mathsf{E}\left[\sum_{i=1}^{n} h_i IN_i + (b + h'_1)B\right]$$

Both are equivalent.

Derivation of optimal policy (1)

$\bar{C}_i(x)$: expected cost for 1 to i points when IN_{i+1} is x

$\hat{C}_i(x)$: expected cost for 1 to i points when IN_i is x

$C_i(y)$: expected cost for 1 to i points when ITP_i is y

Initial condition:

$$\bar{C}_0(x) = (b + h_1')[x]^-$$

Derivation of optimal policy (2)

Expected cost for 1 to i inventory points when IN_i is x:

$$\hat{C}_i(x) = h_i x + \bar{C}_{i-1}(x)$$

Expected cost for 1 to i inventory points when ITP_i is y:

$$C_i(y) = \mathsf{E}\left[\hat{C}_i(y - D_i)\right]$$

Expected cost for 1 to i inventory points when IN_i is x:

$$\bar{C}_i(x) = C_i(\min\{s_i^*, x\})$$

Derivation of optimal policy (3)

Optimal echelon base stock level s_i^*:

$$s_i^* = \arg\min C_i(y)$$

Since the echelon base stock level must be nonincreasing,

$$s_i^{-*} = \min_{i \leq j} s_j^*$$

The optimal local policy $s_i'^*$:

$$s_i'^* = s_i^{-*} - s_{i-1}^{-*}$$

where s_0^{-*} is 0.

Periodic ordering policy

Single stage model

D_t: demand in period t

L: leadtime from production to inventory

s: base-stock level

c: production capacity

q_t: production or ordering amount in period t

$$q_t = \min\left\{c, [s - (I_t - D_t + T_t)]^+\right\}$$

Periodic ordering system

I_t: net inventory (inventory $-$ backlog) in period t:

$$I_{t+1} = I_t - D_t + q_{t-L}$$

T_t: pipeline or in-transit inventory in period t:

$$T_{t+1} = T_t + q_t - q_{t-L}$$

h: inventory holding cost

b: lost sales (backlog) cost

Expected cost

Expected cost C_t in period t:

$$C_t = b[I_t]^- + h[I_t]^+$$

Objective function is expectation over t_{max} periods:

$$\frac{1}{t_{max}} \sum_{t=1}^{t_{max}} \mathsf{E}[C_t]$$

$\Rightarrow$ calculate the derivative of the objective function w.r.t. state variable s

Derivative recursions

$$\frac{dI_{t+1}}{ds} = \frac{I_t}{ds} + \frac{dq_{t-L}}{ds}$$

$$\frac{dT_{t+1}}{ds} = \frac{dT_t}{ds} + \frac{dq_t}{ds} - \frac{dq_{t-L}}{ds}$$

$$\frac{dq_t}{ds} = \begin{cases} 0 & \text{if capacity bound} \\ 1 - \left(\frac{dI_t}{ds} + \frac{dT_t}{ds}\right) & \text{otherwise} \end{cases}$$

Initial inventory: $I_0 = s$
$\Rightarrow$ initial derivative of inventory: $I_0' = 1$
other derivatives: $(T_0)', (q_0)' = 0$

Derivatives of C_t

$$(C_t)' = -b(I_t)'\mathbf{1}[I_t < 0] + h(I_t)'\mathbf{1}[I_t > 0]$$

Derivative of the expectation of C_t of s converges to

$$\mathsf{E}\left[\frac{1}{t_{max}}\sum_{t=1}^{t_{max}}(C_t)'\right]$$

with probability 1.

Series periodic model

Echelon base-stock policy:
Givenechelon base-stock level for the i-th point s^i, try to restore the echelon inventory position

$$\sum_{j=1}^{i}(I_t^j + T_t^j) - D_t$$

to s_i.
Thus, we order:

$$q_t^i = \min\left\{c_i, \left[s^i + D_t - \sum_{j=1}^{i}(I_t^j + T_t^j)\right]^+, [I_t^{i+1}]^+\right\}.$$

Recursions

$$I_{t+1}^{i} = I_{t}^{i} - q_{t}^{i-1} + q_{t-L_i}^{i}$$

$$T_{t+1}^{i} = T_{t}^{i} + q_{t}^{i} - q_{t-L_i}^{i}$$

Initial inventory: $I_0^1 = s^1, \;\; I_0^i = s^i - s^{i-1}, i = 2, \cdots, m$
All other variables are set to 0

Cost and expectation

Cost in period t:

$$C_t = b[I_t^1]^- + h^1[I_t^1]^+ + \sum_{j=2}^{m} h^j(I_t^j + T_t^{j-1})$$

where b is the backlog cost and h^i is the inventory cost at the i-th inventory point.
Expected cost:

$$\frac{1}{t_{max}} \sum_{t=1}^{t_{max}} \mathsf{E}[C_t]$$

Derivative recursions

$$\frac{dq_t^i}{ds^*} = \begin{cases} 0 & \text{if } i \text{ is capacity bound} \\ 0 & \text{if order amount is } 0 \\ (I_t^{i+1})' & \text{if } i \text{ is supply bound} \\ \mathbf{1}[i = i^*] - \sum_{j=1}^{i} \left(\frac{dI_t^j}{ds^*} + \frac{dT_t^j}{ds^*} \right) & \text{otherwise} \end{cases}$$

$$\frac{dI_{t+1}^i}{ds^*} = \frac{I_t^i}{ds^*} - \frac{dq_t^{i-1}}{ds^*} + \frac{dq_{t-L_i}^i}{ds^*}$$

$$\frac{dT_{t+1}^i}{ds^*} = \frac{dT_t^i}{ds^*} + \frac{dq_t^i}{ds^*} - \frac{dq_{t-L_i}^i}{ds^*}$$

Derivative of C_t

Derivative in period t:

$$(C_t)' = -b(I_t^1)'\mathbf{1}[I_t^1 < 0] + h_1(I_t^1)'\mathbf{1}[I_t^1 > 0] + \sum_{i=2}^{m} h_i \left\{(T_t^i)' + (I_t^{i-1})'\right\}$$

Expectation:

$$\mathsf{E}\left[\frac{1}{t_{max}} \sum_{t=1}^{t_{max}} (C_t)'\right]$$

Robust optimization approach

I_t: net inventory in period t

$$I_{t+1} = I_t - D_t + q_{t-L}$$

$$I_{t+1} = I_0 + \sum_{k=0}^{t} (q_{k-L} - D_k)$$

inventory cost h, backlog cost b, expected cost C_t

$$C_t = b[I_t]^- + h[I_t]^+$$

Constraints:

$$C_t \geq hI_t$$

$$C_t \geq -bI_t$$

Formulation

Ordering fixed cost K, 0-1 variable ξ_t, a big number M

$$
\begin{array}{lll}
\text{minimize} & \displaystyle\sum_{t=1}^{t_{max}} (K\xi_t + C_t) & \\
\text{subject to} & C_t \geq h\left\{I_0 + \displaystyle\sum_{k=0}^{t}(q_{k-L} - D_k)\right\} & \forall t \\
& C_t \geq -b\left\{I_0 + \displaystyle\sum_{k=0}^{t}(q_{k-L} - D_k)\right\} & \forall t \\
& 0 \leq q_t \leq M\xi_t & \forall t \\
& \xi_t \in \{0, 1\} & \forall t
\end{array}
$$

Robust optimization approach

$D_t \in [\bar{D}_t - \hat{D}_t, \bar{D}_t + \hat{D}_t]$: demand (random variable in an interval)

Γ_t : buget of uncertanity

auxialy variables $0 \le z_{kt} \le 1$

$$C_t \ge h \left\{ I_0 + \sum_{k=0}^{t} (q_{k-L} - \bar{D}_k) + \max_{\sum_k z_{kt} \le \Gamma_t} \sum_{k=0}^{t} \hat{D}_k z_{kt} \right\}$$

y_{kt}: Dual variables for the constraints $z_{kt} \le 1$

θ_t : Dual variable for the budget constraint $\sum_{k=0}^{t} z_{kt} \le \Gamma_t$

Robust model

$$\begin{aligned}
\text{minimize} \quad & \sum_{t=1}^{t_{max}} (K\xi_t + C_t) \\
\text{subject to} \quad & C_t \geq h \left\{ I_0 + \sum_{k=0}^{t} \left(q_{k-L} - \bar{D}_k\right) + \theta_t \Gamma_t + \sum_{k=0}^{t} y_{kt} \right\} && \forall t \\
& C_t \geq -b \left\{ I_0 + \sum_{k=0}^{t} \left(q_{k-L} - \bar{D}_k\right) - \theta_t \Gamma_t - \sum_{k=0}^{t} y_{kt} \right\} && \forall t \\
& \theta_t + y_{kt} \geq \hat{D}_k && \forall k, \\
& 0 \leq q_t \leq M\xi_t && \forall t \\
& \xi_t \in \{0, 1\} && \forall t
\end{aligned}$$